Check out the interactive Riemann Sums demo available on Wolfram Demonstrations:
https://demonstrations.wolfram.com/RiemannSums/

→ quiz 1 on Thursday
→ Turning Point poll by 12:45

Section 5.5: Integration by substitution

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Today's Learning Goals

- Evaluate integrals using the substitution (usub) method
- Understand how to choose u
- Understand which functions can be evaluated with the substitution method
- The substitution method is a *change of variable* in the integral that simplifies the integrand into **f(u) du** for a function **f** we recognize

Functions we already know how to integrate directly:

Recall the antiderivatives of the following functions we reviewed last week:

$$x^n, \sin(ax), \cos(ax)$$

$$\csc(ax)\cot(ax)$$

$$\sec(ax)\tan(ax)$$

$$\sec^2(ax), \csc^2(ax)$$

$$e^{ax}, b^{ax}$$

$$\frac{1}{1+(ax)^2}, \frac{1}{\sqrt{1-(ax)^2}}$$

(know these, or understand how to get the antiderivative formulas)

Method of u-substitution

This method is the reverse of the chain rule for derivatives:

Let F be an antiderivative of f. Let $u = g(x)$.

$$\int f(g(x))\cdot g'(x)dx = \int f(u)du = F(u) + C$$

In other words:

$$\int f(stuff)\cdot(stuff)'dx = F(stuff) + C$$

u-substitution with Definite Integrals

To evaluate $\int_a^b f(g(x))g'(x)dx$,

set $u = g(x)$ and *change the limits of integration* to match the new variable:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 1.1: Evaluate.

$$I = \int \frac{\cos\left(\sqrt{t}\right)}{\boxed{\sqrt{t}}\sin\left(\sqrt{t}\right)}\boxed{dt}$$

→ what do we choose as u? why?

→ first take $u = \sqrt{t}$, $du = \frac{1}{2\sqrt{t}}dt$

$$I = 2\int \frac{\cos(u)}{\sin(u)}du$$

→ another u-sub

$v = \sin(u)$, $dv = \cos(u)du$

$$I = 2\int \frac{dv}{v}$$

$= 2\ln|v| + C$

$= 2\ln|\sin u| + C$

→ In total: $I = 2\ln|\sin(\sqrt{t})| + C$

Example 1.2: Evaluate. $I = \int \frac{dx}{x(\ln x)^3}$

$\to$ what to choose as u?

$u = \ln x,\ du = \frac{dx}{x}$

$\to I = \int \frac{du}{u^3}$

$= -\frac{1}{2u^2} + C$

$= -\frac{1}{2(\ln x)^2} + C$

Example 1.3: Evaluate $I = \int w\sqrt{1+w}\,dw$

$\to$ what to choose as u?

$u = 1+w,\ du = dw,\ w = u-1$

$\to I = \int (u-1)u^{1/2}\,du$

$= \int (u^{3/2} - u^{1/2})\,du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$I = \frac{2}{5}(1+w)^{5/2} - \frac{2}{3}(1+w)^{3/2} + C$

$$I = \int w\sqrt{1+w}\, dw$$

what if we choose $u = w$, $du = dw$,

$1+w = u+1$

$$I = \int u\sqrt{u+1}\, du$$

Example 2: Evaluate the integral.

$$\int (\sin 6x) e^{\cos 6x}\, dx = I$$

(A) $\frac{1}{6} e^{\cos 6x} + C$

$\boxed{\text{(B) } -\frac{1}{6} e^{\cos 6x} + C}$

(C) $\frac{1}{6} (\cos 6x) e^{\cos 6x} + C$

(D) $\frac{1}{2} \left(e^{\cos 6x}\right)^2 + C$

$\rightarrow$ two clear choices:

* ① $u = \cos(6x)$

② $u = \sin(6x)$

$\rightarrow u = \cos(6x),\ du = -6 \cdot \sin(6x)\, dx$

$\leftrightarrow -\frac{1}{6} du = \sin(6x)\, dx$

$\rightarrow I = -\frac{1}{6} \int e^u\, du = -\frac{1}{6} e^{\cos 6x} + C$

Example 3.2:

Evaluate the following indefinite integral: $I = \int \tan(x)dx$

$$\rightarrow I = \int \frac{\sin x}{\cos x} dx$$

$$u = \cos x, \; du = -\sin x \cdot dx$$

$$\rightarrow I = -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

$$-\ln(a) = \ln\left(\frac{1}{a}\right)$$

Example 3.1:

Evaluate the following indefinite integral: $I = \int \sec(x)dx$

Hint:
Take

$$u = \sec x + \tan x$$

to get that

$$\sec x = \frac{u'}{u}$$

(logarithmic derivative)

$u = \sec x + \tan x$

$du = \sec x \cdot \tan x + \sec^2 x$

$= (\sec x + \tan x)\sec x$

$du \equiv u'$

get that $\sec x = \frac{du}{u} \equiv \frac{u'}{u}$

logarithmic derivative of f:

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)}$$

$$\sec x = \frac{u'(x)}{u(x)} = \frac{d}{dx}\left[\ln u(x)\right]$$

$\rightarrow$ integrate both sides

$$I = \int \sec x \, dx = \ln|u(x)| + C$$

$$= \ln|\sec x + \tan x| + C$$

Additional Trig Formulas (know how to derive these):

$$\int \tan(u)\,du = \ln|\sec u| + C$$

$$\int \sec(u)\,du = \ln|\sec u + \tan u| + C$$

worked these today

$$\int \cot(u)\,du = \ln|\sin u| + C$$

$$\int \csc(u)\,du = -\ln|\csc u + \cot u| + C$$

Work these on your own

Extra problems (limits of integration)

Evaluate the following indefinite integral: $I = \int_0^{\sqrt{\frac{\pi}{4}}} x\cos(x^2)\,dx$

→ what to choose as u?

$u = x^2 \quad \frac{1}{2}du = x\,dx \qquad du = 2x\,dx$

$\rightarrow I = \frac{1}{2}\int_0^{\pi/4} \cos(u)\,du$

$= \frac{1}{2}\left(\sin u\right)\Big|_0^{\pi/4}$ (apply FTC)

$$= \frac{1}{2}\left(\sin(\pi/4) - \sin(0)\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{2}}{2} - 0\right) = \frac{\sqrt{2}}{4}$$

Challenge problem (foreshadowing trig subs – later)

Hints:

1. See that

$$\cos(u) = \sqrt{1 - \sin^2(u)}, u \geq 0$$

2. Write

$$x = \sin(u),$$
$$dx = \cos(u)du$$

1. Use the identity

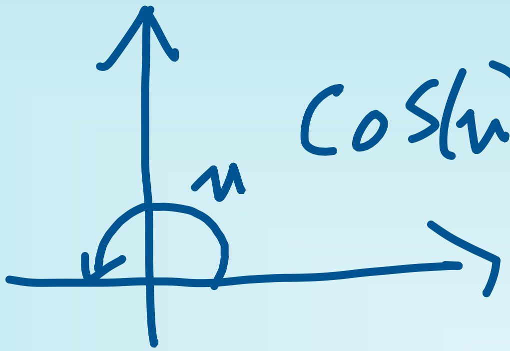
$$\cos^2(u) = \frac{1}{2}(1 + \cos(2u))$$

Evaluate the following indefinite integral: $I = \int_0^1 \sqrt{1 - x^2}dx$

$\underline{\text{Hint 1}}$: $\sin^2 u + \cos^2 u = 1$

$\rightarrow \cos u = \pm\sqrt{1 - \sin^2 u}$

when $u \geq 0$,



$\cos(u) \geq 0$

$\cos u = \sqrt{1 - \sin^2 u}$ (*)

Hint2: write $x = \sin u$, $u = \sin^{-1}(x)$

$$dx = \cos u \, du$$

$$I = \int_{\sin^{-1}(0)}^{\sin^{-1}(1)} \sqrt{1-\sin^2 u} \cdot \cos(u)\, du$$

$\sin^{-1}(1) \to \pi/2$

$\sin^{-1}(0) \to 0$

$$= \int_0^{\pi/2} \cos^2(u)\, du$$

$$I = \int_0^1 \sqrt{1-x^2}\, dx$$

Hint 3: $\cos^2(u) = \frac{1}{2}\left(1+\cos(2u)\right)$

$$I = \frac{1}{2}\int_0^{\pi/2}\left(1+\cos(2u)\right)du$$

$$= \frac{1}{2}\left(u + \frac{1}{2}\sin(2u)\right)\Big|_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\pi/2 - 0\right) + \frac{1}{2}\left(\sin(\pi) - \sin(0)\right)\right]$$

$$\boxed{= \pi/4}$$

Section 5.6: Area between two curves

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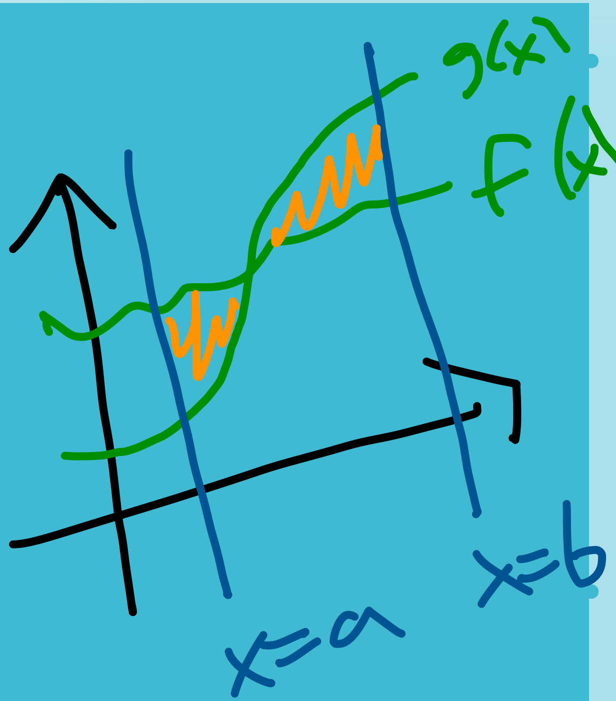
Today's Learning Goals

- Understand what is meant graphically by integrating the difference between two functions (*solve for intersection points between the two curves on the interval*)
- Set up an integral to find the total area bounded between two curves
- Evaluate numerically the area bounded between two curves
- Be able to express the integration in terms of either x or y, depending on the function(s)

Area Between Two Curves

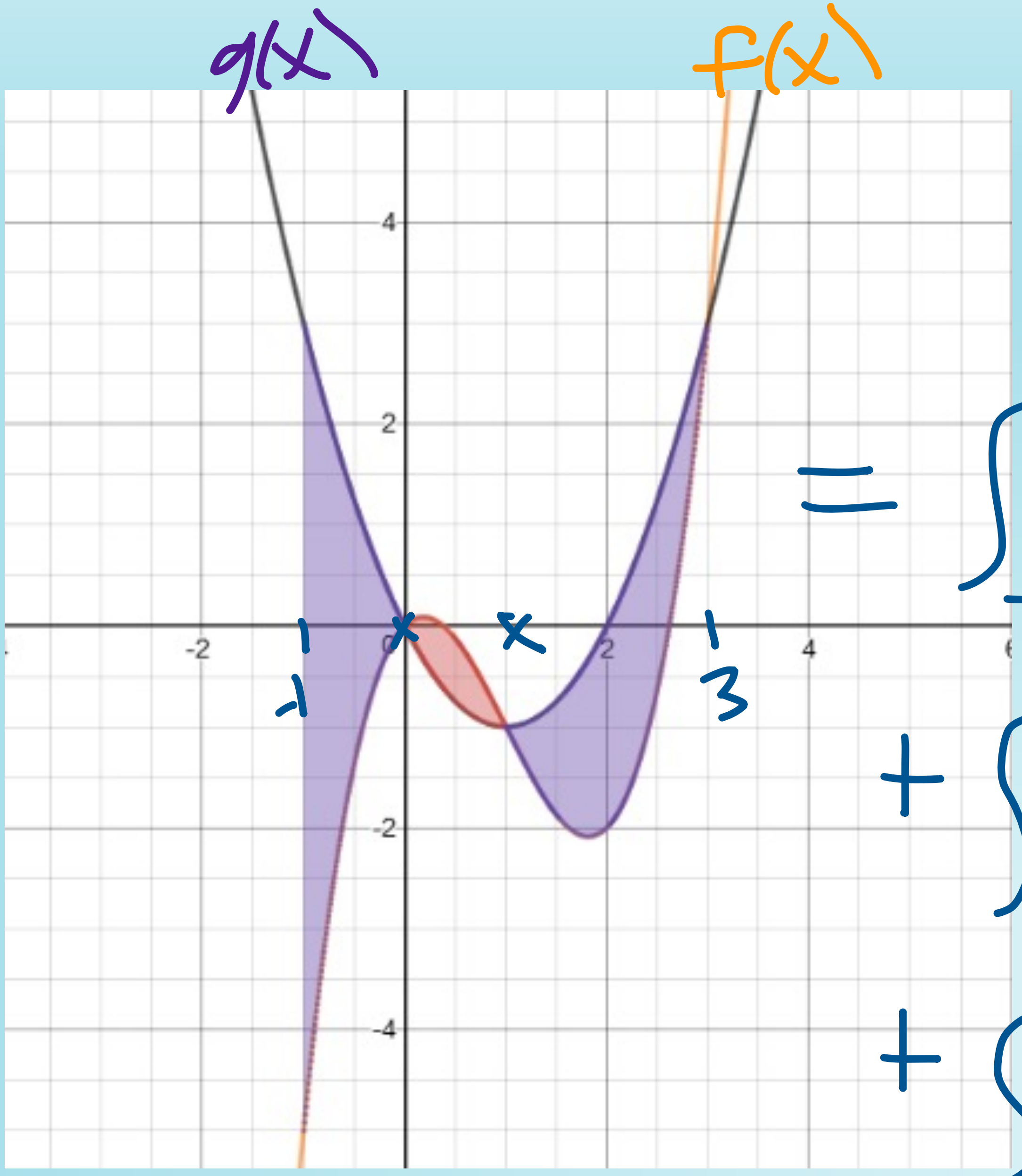


- To find the area between two curves, written as functions of *x*:

$$A = \int_a^b | f(x) - g(x) | \, dx = \int_a^b (top - bottom) dx$$

- To find the area between two curves, written as functions of *y*:

$$A = \int_a^b | f(y) - g(y) | \, dy = \int_a^b (right - left) dy$$



$$\int_{-1}^{3} |f(x)-g(x)|\,dx$$

$$= \int_{-1}^{0} (g(x)-f(x))\,dx$$

$$+ \int_{0}^{1} (f(x)-g(x))\,dx$$

$$+ \int_{1}^{3} (g(x)-f(x))\,dx$$